

Electric-Magnetic Duality in Massless QED?

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Abstract

The possibility that QED and recently developed non-Hermitian, or magnetic, versions of QED are equivalent is considered. Under this duality the Hamiltonians and anomalous axial currents of the two theories are identified. A consequence of such a duality is that particles described by QED carry magnetic as well as electric charges. The proposal requires a vanishing zero bare fermion mass in both theories; Dirac mass terms are incompatible with the conservation of magnetic charge much as Majorana masses spoil the conservation of electric charge. The physical spectrum comprises photons and massless spin- $\frac{1}{2}$ particles carrying equal or opposite electric and magnetic charges. The four particle states described by the Dirac fermion correspond to the four possible charge assignments of elementary dyons. This scale invariant spectrum indicates that the quantum field theory is finite. The Johnson Baker Willey eigenvalue equation for the fine structure constant in finite spinor QED is interpreted as a Dirac-like charge quantisation condition for dyons.

1 Introduction

In the past few years it has been demonstrated that some very simple non-Hermitian Hamiltonians provide well-defined quantum theories [1, 2]. These theories are related via (non-unitary) similarity transformations [3] to ‘standard’ Hermitian quantum theories. In most cases it is difficult to determine the form of the equivalent Hermitian theory. In this paper a non-Hermitian form of quantum electrodynamics (QED) is considered. A non-hermitian, but \mathcal{PT} -symmetric, form of QED was proposed by Bender and Milton [4]. Later, Milton [5] argued that, due to an axial anomaly, the theory is not renormalisable. However, he suggested an alternative non-Hermitian theory using an axial vector field instead of an axial current. Milton’s theory was further developed in reference [6]. The present author [7] has argued that this theory is a magnetic form of QED in that it describes the interactions of spin $\frac{1}{2}$ particles carrying magnetic rather than electric charges. It is not clear what is the Hermitian field theory (or theories) that is equivalent to magnetic QED (MQED). In this paper we examine the possibility that MQED is actually equivalent to QED so that the similarity transformation between the Hermitian and non-Hermitian descriptions is a form of electric-magnetic duality. This proposal requires a vanishing bare fermion mass in both theories. This is because Dirac masses are incompatible with the conservation of magnetic charge much as Majorana masses spoil the conservation of electric charge. Both versions of QED have the same set of discrete symmetries \mathcal{P} , \mathcal{C} and \mathcal{T} and their anomalous axial currents can be identified. The conserved currents associated with the global gauge symmetries of the two theories are, however, not identified. The electric current of QED can be expressed locally in terms of the Dirac fermion fields whereas the magnetic current of MQED is local in the fermion fields of the non-Hermitian theory. It is argued that both conserved currents are present in both theories but the magnetic current is non-local in standard QED and the electric current is non-local in MQED (though the non-conserved axial current is local in both descriptions). Consequently, the particles described by massless QED (and MQED) carry both electric and magnetic charge. The four particles described by a Dirac fermion correspond to the four possible electric and magnetic charge assignments of elementary dyons.

Assuming that QED and MQED are equivalent and that the physical particles correspond to the field content yields a scale invariant spectrum. The idea of a scale invariant

version of spinor QED has been developed by Johnson, Baker and Willey (JBW) [8, 9]. On the basis of a detailed examination of the ultraviolet and infrared properties of Feynman diagrams entering the photon propagator JBW argued that for special values of the fine structure constant, α , massless QED could be finite and hence scale invariant. Here finite theories correspond to solutions of an ‘eigenvalue’ condition involving the bare fine structure constant. Unfortunately, it is still not known whether this equation has non-trivial solutions. Even if it does, a physical interpretation of any finite theory is lacking. However, Adler [10] has observed that a solution of the eigenvalue equation would provide a finite QED independent of the number of fermion species. Consequently, if the eigenvalue equation has a single non-trivial solution then finite spinor QED incorporates the quantisation of electric charge. It is argued that this is a reflection of the dyonic nature of the physical states; the JBW eigenvalue equation is interpreted as a version of Dirac’s charge quantisation condition.

The outline of this paper is as follows. Non-Hermitian QED is reviewed in the next section. The idea of electric-magnetic duality in QED is outlined in section 3. In section 4 it is explained how the JBW constraint on the fine structure constant can be interpreted as a Dirac-like quantisation condition on the electric and magnetic charges of the dyons. Section 5 includes concluding remarks and some speculation concerning spin-0 dyons.

2 Magnetic QED

Massless QED is based on the Lagrangian (metric and Dirac matrix conventions are as in Bjorken and Drell [11])

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + e\bar{\psi}\gamma^\mu A_\mu\psi, \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Here A_μ is a $U(1)$ gauge potential and ψ is a Dirac spinor. The corresponding quantum theory has a Hermitian Hamiltonian and is symmetric under parity \mathcal{P} and time-reversal \mathcal{T} . Milton considered the Lagrangian [5]¹

$$\mathcal{L} = -\frac{1}{4}G_{\mu\nu}G^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial_\mu\psi + ig\bar{\psi}\gamma^\mu B_\mu\psi, \quad (2)$$

¹ In [5] a real representation for Dirac spinors was adopted. In this paper, as also in [6], a conventional complex representation is assumed.

with $G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, B_μ being an abelian gauge potential, ψ a Dirac spinor and g a real coupling constant. The theory couples a gauge potential B_μ (assumed to be Hermitian) to the anti-Hermitian current $j_\mu = ig\bar{\psi}\gamma_\mu\psi$ which renders the Hamiltonian non-Hermitian. The gauge potential transforms in the usual way under time reversal, i.e.

$$\mathcal{T}B_0(\mathbf{r}, t)\mathcal{T}^{-1} = B_0(\mathbf{r}, -t), \quad \mathcal{T}\mathbf{B}(\mathbf{r}, t)\mathcal{T}^{-1} = -\mathbf{B}(\mathbf{r}, -t), \quad (3)$$

or in a more compact form

$$\mathcal{T}B_\mu(\mathbf{r}, t)\mathcal{T}^{-1} = B^\mu(\mathbf{r}, -t). \quad (4)$$

Under parity a non-standard (pseudovector) transformation is assumed

$$\mathcal{P}B_\mu(\mathbf{r}, t)\mathcal{P}^{-1} = -B^\mu(-\mathbf{r}, t). \quad (5)$$

The resulting theory is non-Hermitian and \mathcal{P} and \mathcal{T} are not symmetries. However, the combined operation \mathcal{PT} is a symmetry and on this basis the theory is expected to have a real spectrum.

An alternative Lagrangian for non-Hermitian QED was given in [7]. This theory has the same set of discrete symmetries as standard QED and thus appears to possess more discrete symmetry than Milton's theory. This is misleading since the two non-Hermitian theories are related by a canonical transformation. The mismatch reflects the use of different parity operators; the two parity operators have the same effect on pure photon states but on fermionic states the Milton \mathcal{P} is equivalent to the \mathcal{CP} operator of [7]. The two theories also 'disagree' with respect to time-reversal. However, it is a feature of this anti-unitary operator that a \mathcal{T} -symmetric theory can be canonically equivalent to a non \mathcal{T} -symmetric theory (a simple example being the Hamiltonian $H(q, p) = q$ which is canonically equivalent to $K(Q, P) = P$; the former is \mathcal{T} -symmetric but the latter is not). The alternative Lagrangian reads

$$\mathcal{L} = -\frac{1}{4}H_{\mu\nu}H^{\mu\nu} + i\bar{\lambda}\gamma^\mu\partial_\mu\lambda + ig\bar{\lambda}\gamma^\mu V_\mu\lambda. \quad (6)$$

Here $H_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ where the gauge potential, V_μ , has unconventional transformations under both \mathcal{T} and \mathcal{P} , that is

$$\mathcal{T}V_\mu(\mathbf{r}, t)\mathcal{T}^{-1} = -V^\mu(\mathbf{r}, -t), \quad \mathcal{P}V_\mu(\mathbf{r}, t)\mathcal{P}^{-1} = -V^\mu(-\mathbf{r}, t). \quad (7)$$

The spinor field λ transforms like a Dirac spinor under proper Lorentz transformations and time-reversal. Under parity it transforms as

$$\mathcal{P}\lambda_\alpha(\mathbf{r}, t)\mathcal{P}^{-1} = P_{\alpha\beta}\lambda_\beta^\dagger(-\mathbf{r}, t), \quad (8)$$

where $P_{\alpha\beta}$ denotes the matrix elements of the Dirac matrix $i\gamma^0\gamma^2$ (here it is assumed that $\gamma_0 = \gamma_0^T$ and $\gamma_2 = \gamma_2^T$). This is actually the standard form of the \mathcal{CP} transformation for Dirac spinor fields. The theory couples the Hermitian gauge potential, V_μ , to the current

$$k_\mu = ig\bar{\lambda}\gamma_\mu\lambda. \quad (9)$$

Under \mathcal{T} and \mathcal{P}

$$\mathcal{T}^{-1}k_\mu(\mathbf{r}, t)\mathcal{T} = -k^\mu(\mathbf{r}, -t), \quad \mathcal{P}^{-1}k_\mu(\mathbf{r}, t)\mathcal{P} = -k^\mu(-\mathbf{r}, t). \quad (10)$$

This non-Hermitian theory is symmetric under \mathcal{T} and \mathcal{P} ; the non-standard transformation properties of V_μ compensate for those of k_μ . The field strength, $H_{\mu\nu}$, transforms like the Maxwell dual field strength and satisfies the \mathcal{P} and \mathcal{T} symmetric equation of motion

$$\partial_\mu H^{\mu\nu} = k^\nu. \quad (11)$$

A gauge-invariant mass term

$$\mathcal{L}_{mass} = -m\bar{\lambda}\lambda, \quad (12)$$

may be added to the Lagrangian (6). This looks like a Dirac mass term but *physically* it is a Majorana mass. Viewed as perturbations, Dirac and Majorana masses allow a massless fermion to transform into its \mathcal{P} and \mathcal{CP} conjugate, respectively. Due to the switched transformation properties of the λ -spinor under \mathcal{P} and \mathcal{CP} a spin- $\frac{1}{2}$ monopole can have a Majorana mass but not a Dirac mass. This complements the well known result that a spin- $\frac{1}{2}$ particle carrying electric charge may have a Dirac mass but not a Majorana mass.

Returning to the massless theory, the Lagrangian (6) possesses the global symmetry

$$\lambda \rightarrow e^{s\gamma_5}\lambda, \quad \bar{\lambda} \rightarrow \bar{\lambda}e^{s\gamma_5}, \quad (13)$$

where s is a constant. This symmetry implies that the current

$$k_\mu^5 = \bar{\lambda}\gamma_\mu\gamma_5\lambda, \quad (14)$$

is conserved at the classical level. Quantum mechanically this current exhibits an anomalous divergence²

$$\partial^\mu k_\mu^5 = -\frac{g^2}{8\pi^2} H^{\mu\nu} \tilde{H}_{\mu\nu}, \quad (15)$$

where $\tilde{H}^{\mu\nu}$ is the Hodge dual of $H^{\mu\nu}$.

Although MQED has a non-Hermitian Hamiltonian it is expected that it can be brought into to a Hermitian form via a (non-unitary) similarity transformation

$$S^{-1} H_{MQED} S = h \quad \text{with} \quad h^\dagger = h. \quad (16)$$

The Dirac inner-product $(\psi, \phi)_D = \langle \psi | \phi \rangle$ is not invariant under such similarity transformations. The choice of the Dirac inner-product within the hermitian theory is equivalent to a modified inner-product for the non-Hermitian theory

$$(\psi, \phi) = \langle \psi | \eta | \phi \rangle, \quad (17)$$

where $\eta = (S^\dagger S)^{-1}$. Instead of focussing on the similarity transformation, S , one may take (17) as a starting point, with a view to determining a suitable η operator. In particular, the Dirac inner product does not provide unitary time-evolution for a non-Hermitian Hamiltonian. The inner-product (17) does provide this if

$$\eta H - H^\dagger \eta = 0. \quad (18)$$

In general, if U belongs to the symmetry group of the theory (including discrete as well as continuous symmetries) one has

$$U^\dagger \eta U = \eta. \quad (19)$$

Equation (18) is (19) applied to time translations. In MQED, as well as other non-Hermitian theories, it is straightforward to identify an operator η fulfilling equation (18). Unfortunately, a solution to equation (18) on its own is not sufficient to guarantee that the associated inner-product is the physical one. In reference [6] a perturbative expansion for η is developed.

² This is a formal continuation of the QED anomaly equation with $F_{\mu\nu}$ replaced with $H_{\mu\nu}$ and e replaced with ig .

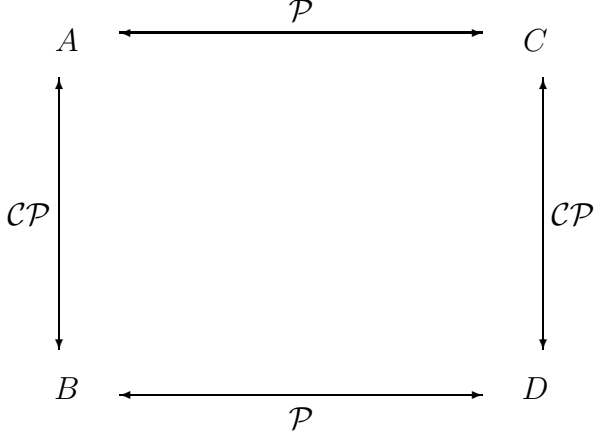


Figure 1: The four particles are related by the discrete symmetries \mathcal{P} and \mathcal{CP}

3 Electric-Magnetic Duality in QED

Assuming that the fields in QED correspond to the physical particles, the spectrum is based on photons and four fermions derived from the Dirac spinor field ψ . In massless QED the one fermion states comprise left and right handed electrons and their antiparticles:

- A) left-handed electron
- B) right-handed anti-electron
- C) right-handed electron
- D) left-handed anti-electron

These particles can be transformed into each other through the discrete symmetries \mathcal{P} , \mathcal{CP} and \mathcal{C} .

Now assume the existence of (non-negative) commuting number operators N^A , N^B , N^C and N^D for the particle types listed above. Under \mathcal{P} and \mathcal{CP} they transform according to Fig.1, for example, $\mathcal{P}^{-1}N^A\mathcal{P} = N^C$. The number operators are invariant under time-reversal, so that $\mathcal{T}^{-1}N^A\mathcal{T} = N^A$, etc. In QED a construction of such operators is lacking (except for the free theory). However, operators are known which can be interpreted as

two particular linear combinations:

$$Q_e = e(N^A - N^B + N^C - N^D), \quad (20)$$

and

$$Q_5 = N^A - N^B - N^C + N^D. \quad (21)$$

The electric charge, Q_e , and axial charge, Q_5 , are defined as volume integrals of the local currents $j_\mu = e\bar{\psi}\gamma_\mu\psi$. and $j_\mu^5 = \bar{\psi}\gamma_\mu\gamma_5\psi$, respectively. Electric charge is conserved whereas the axial current satisfies the anomalous divergence condition [12, 13, 14]

$$\partial^\mu j_\mu^5 = \frac{\alpha}{2\pi} F^{\mu\nu} \tilde{F}_{\mu\nu}, \quad (22)$$

where $\alpha = e^2/4\pi$ is the bare fine structure constant.

Comparing massless QED with massless MQED one has two quantum theories with the same set of discrete symmetries and the same number of spinor and vector degrees of freedom. Can the two theories be equivalent? This would mean that the two Hamiltonians would be related by a similarity transformation of the form (16) with $h = H_{QED}$. If this holds how do the operators or degrees of freedom match? At the operator level it is natural to identify the anomalous currents,

$$S^{-1}k_\mu^5 S = j_\mu^5 \quad (23)$$

since they have the same transformation properties with respect to the discrete symmetries. Moreover, the associated axial charges are expected to have integer eigenvalues. The anomaly equations then give

$$-g^2 S^{-1} H^{\mu\nu} \tilde{H}_{\mu\nu} S = e^2 F^{\mu\nu} \tilde{F}_{\mu\nu}. \quad (24)$$

which is consistent with the identifications

$$S^{-1} H_{\mu\nu} S \stackrel{?}{=} \tilde{F}_{\mu\nu}, \quad (25)$$

and

$$g = \pm e. \quad (26)$$

The identification (25) is too strong to hold for all matrix elements; it is only expected to hold with respect to photon states.³ One can understand this at the classical level. In the presence of magnetic charges $F_{\mu\nu}$ requires Dirac strings whereas for $H_{\mu\nu}$ string singularities are attached to electric charges. Therefore $\tilde{F}_{\mu\nu}$ and $H_{\mu\nu}$ cannot be identical in the presence of charges.

The conserved currents associated with the global gauge symmetries cannot be identified as they transform differently under \mathcal{P} , \mathcal{CP} and \mathcal{T} . For example, under parity

$$\mathcal{P}^{-1}j_\mu(\mathbf{r}, t)\mathcal{P} = j^\mu(-\mathbf{r}, t), \quad \mathcal{P}^{-1}k_\mu(\mathbf{r}, t)\mathcal{P} = -k^\mu(-\mathbf{r}, t). \quad (27)$$

Now consider the charge

$$Q_m = ig(N^A + N^B - N^C - N^D), \quad g = \pm e. \quad (28)$$

This operator cannot be expressed locally in the QED variables. However, it has the same transformation properties as the magnetic charge operator in MQED (defined as the volume integral of k_0). Under the proposed duality massless QED possesses an *additional* conserved current

$$j_\mu^m(x) = S^{-1}k_\mu(x)S. \quad (29)$$

It is natural to identify Q_m as a volume integral of j_μ^0 . Consequently, massless QED has an additional symmetry which has no classical analogue. One can say that this symmetry ‘replaces’ chiral symmetry which, by virtue of the axial anomaly, does not survive quantisation. Although the symmetry groups of classical and quantum electrodynamics are different they have the same dimension.

We have argued that scale invariant QED and MQED describe the same physics. Accordingly, the one particle states carry electric *and* magnetic charge; a state $|A\rangle$ which is an eigenvector of N^A with unit eigenvalue has the properties

$$Q_e|A\rangle = e|A\rangle, \quad Q_m|A\rangle = ig|A\rangle, \quad (30)$$

³ Assuming a finite QED and MQED we have $T^{00} = \frac{1}{2}(E^2 + B^2) + \text{fermion bilinear}$ in both theories. Pure photon states are annihilated by the fermion bilinear. Therefore, on photon states the constant of proportionality between $S^{-1}H_{\mu\nu}S$ and $\tilde{F}_{\mu\nu}$ must be ± 1 which is the basis for (26). A loophole in the argument is that $F^{\mu\nu}\tilde{F}_{\mu\nu}$ annihilates pure photon states.

and similarly for one particle states $|B\rangle$, $|C\rangle$ and $|D\rangle$ which are \mathcal{CP} -, \mathcal{P} - and \mathcal{C} - conjugates of $|A\rangle$, respectively. A simple process which exhibits the conservation of electric and magnetic charge is

$$A + D \rightarrow B + C. \quad (31)$$

Here $\Delta Q_5 = -4$. One can also have annihilation of charge conjugate pairs, e.g.

$$A + D \rightarrow \text{photons}, \quad (32)$$

for which $\Delta Q_5 = -2$.

4 Finite QED and the Quantisation of Electric and Magnetic Charge

An electric-magnetic duality between QED and MQED suggests a spectrum based on photons and massless spin- $\frac{1}{2}$ dyons carrying equal or opposite electric and magnetic charges. The four particle states associated with a Dirac fermion correspond to the four possible charge assignments of elementary dyons. Therefore if this duality is realised the relevant quantum field theory is scale-invariant. The idea that a finite scale-invariant version of spinor QED may exist was advanced by Johnson, Baker and Willey (JBW) in the early 1960s. Starting from an integral representation of the photon propagator JBW argued that QED is finite⁴ if α satisfies the ‘eigenvalue’ condition

$$f(\alpha) = 0, \quad (33)$$

where $f(\alpha)$ is a coefficient in the large-momentum expansion of the integrand. Unfortunately, it is still not known for what positive values of α , if any⁵, equation (33) holds. Although $f(\alpha)$ has been analyzed perturbatively the determination of the roots of f is a non-perturbative problem. Adler [10] has observed that a solution of (33) would provide a finite QED independent of the number of fermion species. In other words if (33) has a single positive solution then finite spinor QED incorporates the quantisation of electric

⁴ The possibility of a finite photon propagator was considered earlier by Gell-Mann and Low [15]. JBW went further by positing the finiteness of full QED.

⁵It has been suggested that (33) has no positive solution, see for example [16].

charge. We would like to interpret this quantisation condition, and hence the JBW eigenvalue equation, as a Dirac-like quantisation condition on the electric and magnetic charges of the dyons. Dirac's original quantisation condition [17], constraining the charge of an electron interacting with a monopole, does not apply to dyons. Schwinger and Zwanziger (SZ) have argued that the possible electric and magnetic charges of any charged particles are constrained by the quantisation condition [18, 19]

$$e_i g_j - g_i e_j = 2\pi n_{ij}, \quad (34)$$

where e_i and g_i denotes the electric and magnetic charge of the i th particle, respectively. n_{ij} is an even integer⁶. Applying this quantisation condition to the proposed dyon spectrum of QED yields

$$\alpha = \frac{1}{2}, \quad (35)$$

assuming that the n_{ij} are restricted to the minimal values 2, -2 and 0. While this may turn out to be the 'correct' value of the fine structure constant, the use of the SZ quantisation condition is questionable in this instance (see also [20]). Basic assumptions underlying the SZ derivation of the dyon quantisation condition are contrary to those of this paper. These differences can be summarised as follows:

i) As in this paper SZ consider Lagrangians/Hamiltonians for spin- $\frac{1}{2}$ particles carrying magnetic charge. Both approaches involve a gauge potential, V_μ , with the transformation properties (7). However, the SZ Hamiltonians are Hermitian and also break \mathcal{P} and \mathcal{T} .

ii) To describe dyons SZ employ Lagrangians with two gauge potentials, A_μ and V_μ . As the associated field strengths are dual, $V_\mu(x)$ can be considered as a functional of A_ν or vice versa. Therefore these Lagrangians are non-local. In this paper dyons are interpreted as the states arising from the quantisation of the local QED Lagrangian or the local MQED Lagrangian.

The Hamiltonians considered by SZ are tied to a generalisation of the Lorentz force law proposed by Dirac [21] wherein a particle carrying an electric charge e and magnetic

⁶ Dirac's quantisation condition allows the n_{ij} to be odd. This corresponds to quantising the angular momentum of the electromagnetic fields in units of $\frac{1}{2}\hbar$. As photons are spin-1 particles a restriction to even n_{ij} is natural.

charge g is governed by the equation of motion

$$m \frac{d^2 x_\mu}{d\tau^2} = (e F_{\mu\nu} + g \tilde{F}_{\mu\nu}) \frac{dx^\nu}{d\tau}, \quad (36)$$

where m is the mass of the particle and τ denotes proper-time. The field strength tensor, $F_{\mu\nu}$, and its Hodge dual, $\tilde{F}_{\mu\nu}$, obey the generalised Maxwell equations

$$\partial_\mu F^{\mu\nu} = j^\nu, \quad \partial_\mu \tilde{F}^{\mu\nu} = j_{mag}^\nu. \quad (37)$$

Electric and magnetic charges are defined as integrals (over a volume containing the charge) of j^0 and j_{mag}^0 , respectively. Non-hermitian forms of QED cannot, at least for weak coupling, produce a force law of the form (36); the coupling of an anti-hermitian current to the gauge potential gives rise to an *attractive* force between like charges [5].

Maxwell's equations (37), the Dirac force law and the SZ quantisation condition are invariant under the Heaviside duality rotation

$$\mathbf{E} \rightarrow \cos \theta \mathbf{E} + \sin \theta \mathbf{B}, \quad \mathbf{B} \rightarrow \cos \theta \mathbf{B} - \sin \theta \mathbf{E}, \quad (38)$$

$$j^\mu \rightarrow \cos \theta j^\mu + \sin \theta j_{mag}^\mu, \quad j_{mag}^\mu \rightarrow \cos \theta j_{mag}^\mu - \sin \theta j^\mu, \quad (39)$$

$$e_i \rightarrow \cos \theta e_i + \sin \theta g_i, \quad g_i \rightarrow \cos \theta g_i - \sin \theta e_i, \quad (40)$$

where θ is a constant. It is clear that there is no room for the transformation (40) in finite QED; the electric charges are fixed by the eigenvalue condition. As our approach is based on the idea of electric-magnetic duality it may seem contradictory to reject Heaviside duality. The electric-magnetic duality we are considering is more akin to Olive-Montonen (OM) duality [24]. Here the interactions of dyons, photons and other particles are described by two distinct quantum field theories - one theory admits a perturbative expansion in the electric charge(s) the other a perturbative expansion in the magnetic charge(s) but non-perturbatively they are equivalent. OM duality relations have been conjectured for certain supersymmetric Yang-Mills-Higgs theories. These theories are Hermitian and are also expected to be finite.

If the SZ quantisation condition does not apply here, what should replace it? A straightforward proposal is to determine the roots of f . If there is a single positive root then this statement on its own would suffice to replace (34). Unfortunately, there is not a clear path to tackle equation (33). In the case of the Dirac quantisation condition there are a

number of intuitive derivations [22, 23] which reproduce Dirac's result; using a mixture of classical and quantum ideas quantisation conditions can be deduced without formulating the full quantum theory. These derivations assume massive, or even infinite mass, particles whereas we seek a quantisation condition for massless dyons. Nevertheless, given that a solution of (33) is not within reach, an informal derivation may provide a hint towards a complete solution. An approach to the Dirac quantisation condition is to consider the motion of an electron in the presence of a static monopole [23]; one finds that the change of a component of the orbital angular momentum, during deflection by the monopole, is always $eg/2\pi$. Assuming that this is quantised in units of \hbar gives Dirac's result. To repeat this exercise for dyons one requires a generalisation of the Lorentz force law to accommodate magnetic charges. Using the Dirac force law leads to a SZ-type quantisation rule. However, the Dirac force law does not fit with MQED since it together with (37) gives a repulsive force between like magnetic charges. A generalisation of the Lorentz force law that incorporates an attractive force between like charges is [7]

$$m \frac{d^2 x_\mu}{d\tau^2} = \left(e F_{\mu\nu} - g \tilde{F}_{\mu\nu} \right) \frac{dx^\nu}{d\tau}, \quad (41)$$

where the generalised Maxwell equations (37) are unchanged. A study of the classical dynamics of dyons under this force law might be instructive.

5 Concluding Remarks

In this paper we have considered the possibility of an Olive-Montonen type duality between massless QED and a non-hermitian version of massless QED. The proposal is based on the equivalence of the discrete symmetries and the observation that the properties of the anomalous axial currents match. As the quantum theory is simultaneously described by an electric and magnetic version of QED physical particles carry both electric and magnetic charge. That is, massless QED is a theory of interacting dyons. Accordingly, a Dirac-like quantisation condition is expected. We have argued that this corresponds to the JBW eigenvalue condition developed in studies of finite spinor QED. This interpretation is consistent with Adler's observation concerning species independence.

The interpretation of massless QED as a theory of dyons appears not to extend to scalar electrodynamics - one complex scalar field cannot be identified with four elementary

dyons. This fits with suggestions that the JBW program is not tenable for scalar QED [25, 26]. Does this mean that there is no relativistic quantum theory of abelian spin-0 dyons? There is, however, a conceptually simple though speculative, way to obtain spinless dyons from *spinor* QED. Hitherto we have assumed that one particle states correspond to the field content of the theory. In a strongly coupled theory this is possible but by no means mandatory, e.g. the spectrum of pure Yang-Mills theory does not match its gluon fields. In section 3 it was assumed that one-particle states are eigenvectors of the number operators with one unit and three zero eigenvalues. Instead consider states with eigenvalue $\frac{1}{2}$; a state $|a\rangle$ is assumed to have the properties

$$N^A|a\rangle = \frac{1}{2}|a\rangle, \quad N^B|a\rangle = N^C|a\rangle = N^D|a\rangle = 0. \quad (42)$$

Similarly one can define states $|b\rangle$, $|c\rangle$ and $|d\rangle$ which, much like the one fermion states, are related by the discrete symmetry operations, e.g. $|b\rangle = \mathcal{CP}|a\rangle$. These states carry fractional electric *and* magnetic charge

$$Q_e|a\rangle = \frac{1}{2}e|a\rangle, \quad Q_m|a\rangle = \frac{i}{2}g|a\rangle. \quad (43)$$

The idea that fermionic theories possessing a charge-conjugation symmetry allow fractionally charged states goes back to the work of Jackiw and Rebbi⁷ [27] who argued that such states would be spinless. If this scenario is realised for finite α it is possible that α is not a solution of the eigenvalue condition which derives from the finiteness of the photon propagator. If the charged states are not associated with the fermion fields it is possible that photons are not physical states. Moreover, as the fractionally charged states carry half the electric and magnetic charges of the fields a different value of α may be required to allow a consistent quantisation.

⁷These authors performed a semi-classical analysis of fractionally charged states via fermion zero modes. It is not clear whether the existence of such modes is strictly necessary. In fact, fermion zero modes do exist for certain abelian dyon backgrounds on R^3 [28].

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